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ANALYTICAL THEORY OF THE STRETCH YO-YO FOR DE-SPIN OF SATELLITES

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SUMMARY 14404

An analysis of an advanced type of yo-yo for satellite de-spin is made. The analyzed stretch yo-yo consists of a weight, a spring, a wire, and end fittings, or simply a weight, a spring, and end fittings. Equations of motion are developed for the system but, because of the complex nature of the equations, they are not solved explicitly. By using a novel method of analysis, algebraic complexity is circumvented and simple design equations are derived. A straightforward step by step procedure is given for the design of the stretch yo-yo. The results calculated from the equations clearly indicate that the stretch yo-yo is less sensitive to satellite spin-up errors and uncertainty in the spin moment of inertia than is the conventional rigid yo-yo.

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INTRODUCTION

In the early period of the United States space effort common methods of spin reduction from the initial spin-up of the satellite or the satellite and last stage combination were: retro-rockets, increase of the spin moment of inertia, and, more recently, rigid yo-yo's (References 1 and 2). Invariably there are errors in initial spin-up. These errors can be quite large. For example, in the Explorer XII (1961 υ) satellite launching initial spin-up was about 30 percent greater than the desired spin-up value (189 rpm instead of 150 rpm). This error in spin-up is reflected in the final spin. With retro-rockets the *magnitude* of the error is reflected in the final spin (a 39 rpm increase in nominal initial spin would mean a 39 rpm increase in the final spin). With the change of moment of inertia device or the rigid yo-yo, the *percentage* of the error is reflected in the final spin (a 30 percent increase in nominal initial spin would mean a 30 percent increase in the final spin).

Another source of de-spin error is the uncertainty of the spin axis moment of inertia. Frequently the last rocket stage and the satellite are de-spun together. Because of fuel residue in the last-stage rocket the spin moment of inertia is not accurately known. This variation of inertia from the design value also causes an error in the final spin.

An appreciable error in the final de-spin value can have detrimental effects: Experiments and satellite appendages such as booms and paddles are designed to operate at a certain spin with a modest tolerance about this point. Hence, a large error in de-spin can compromise the experiment(s), and a large de-spin error could cause the satellite appendages not to function or damage them in functioning.

A device that greatly reduces spin-up errors and errors due to variations in spin moment of inertia is the stretch yo-yo (the stretch concept was first suggested by Henry Cornille of Goddard Space Flight Center (Reference 3). It can consist of a weight, a spring, a wire, and end fittings (Figure 1), or simply a weight, a spring, and end fittings (Figure 2). The purpose of the spring is to compensate for errors in initial spin-up. For example, in a given stretch yo-yo application there will be a certain amount of stretching or elongation of the spring during normal operation. If the initial spin is greater than the nominal value, the spring will elongate more than normal during operation and reduce the spin to the desired value. If the initial spin is less than the nominal initial spin, the spring will elongate less

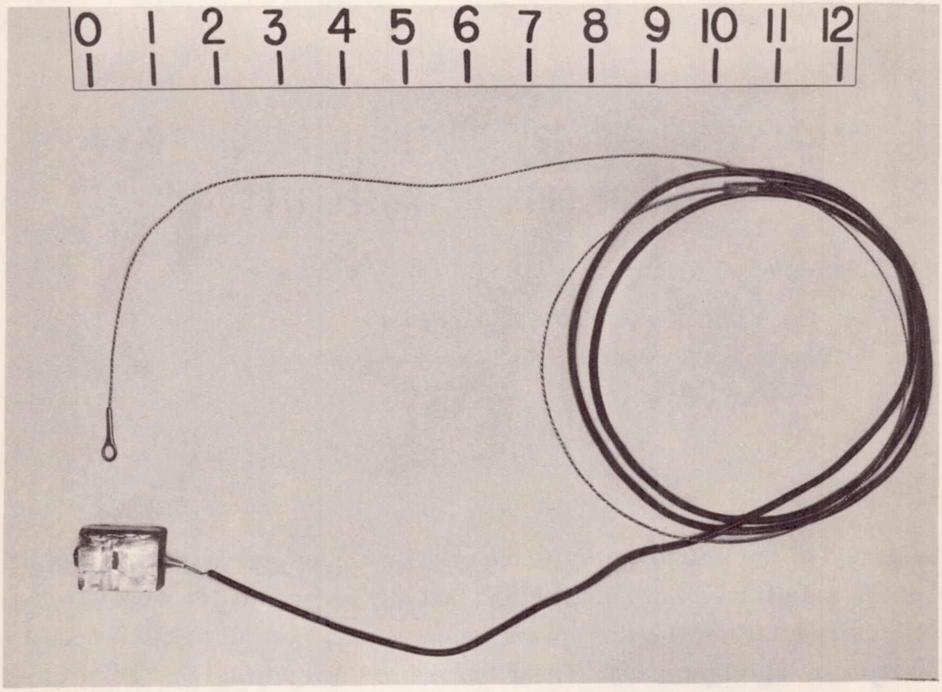

Figure 1—A stretch yo-yo consisting of weight, spring, wire, and end fittings.
The scale is in inches.

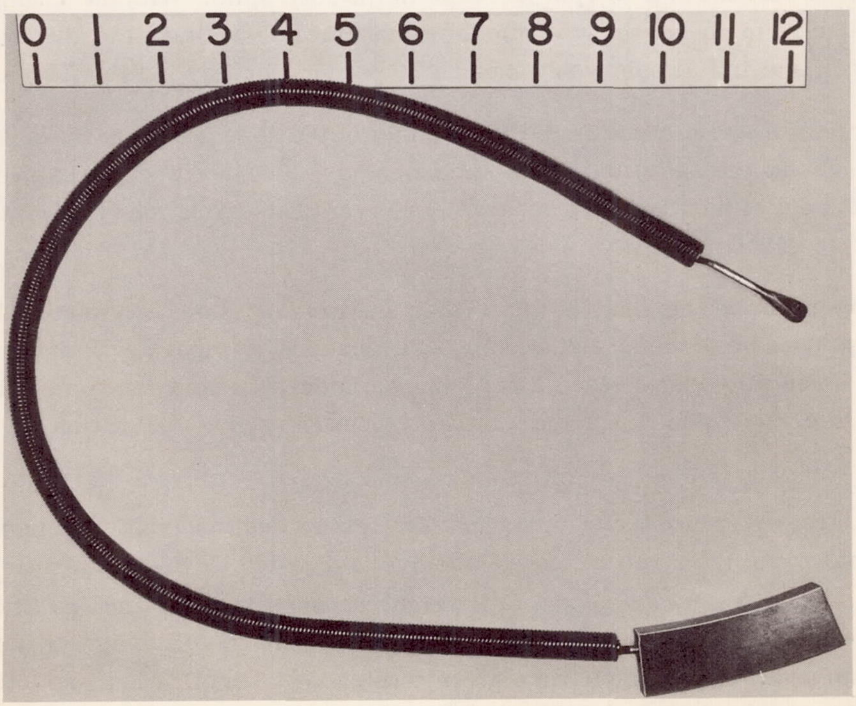

Figure 2—A stretch yo-yo consisting of weight, spring, and end fittings.
The scale is in inches.

than normal during functioning and correct for the under spin. The stretch yo-yo is a simple example of an adaptive control system: it senses the spin environment it is in and corrects accordingly. Also, it will be shown that the stretch yo-yo is relatively insensitive to variations (uncertainties) of the spin moment of inertia.

It should not be concluded that the stretch yo-yo will completely replace the rigid yo-yo for satellite spin reduction. It is believed that the rigid yo-yo still has an application: when de-spin requirements are not stringent, and ease and flexibility of operation are desired (for example, to permit large last minute changes in initial spin-up and spin moment of inertia).

DYNAMIC ANALYSIS

There are essentially two phases to the stretch yo-yo spin reduction process. In Phase 1 the spring changes in length and is tangent to the satellite. In Phase 2 the spring changes its position from tangent to perpendicular to it. At this point (perpendicular to the satellite) the yo-yo is released.

For a given set of satellite parameters and a given spin reduction, the design engineer wants to know the weight of the end mass, to know the proportions of the spring or spring wire, and to verify that there is adequate strength in the spring to insure linear operation of it. As may be expected, analysis of the stretch yo-yo is more complicated than that of the rigid yo-yo. Introducing a spring adds another degree of freedom. Equations of motion will first be developed for Phases 1 and 2, but they will not be solved explicitly because of the complicated nature of the equations. The conservation of momentum and energy equations and a force equation will be applied at the end of Phase 2 deployment, and a novel approach will be used to circumvent the algebraic difficulties encountered in solving these equations.

Phase 1

For analysis purposes, the satellite can be considered to be spinning about a fixed axis but otherwise stationary. A sketch of the Phase 1 coordinate system is shown in Figure 3. Only one spring and weight is shown, as the system is considered symmetrical. Also, the system is considered torque-free; small torques due to the earth's magnetic or gravitational fields, the atmosphere, and the solar sun pressure are neglected. The Lagrangian method of dynamic description will be used to obtain the equations of motion. The total kinetic energy of the system is

$$T = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) , \tag{1}$$

where m is the total mass of both weights, I is the moment of inertia of the satellite about the spin axis, and $\dot{\phi}$, $\dot{x}$, and $\dot{y}$ are velocities. The weight of the springs is taken into account by the method developed in Reference 2. By using the transformation equations,

$$x = a \cos\theta + l \sin\theta , \tag{2}$$

$$y = a \sin\theta - l \cos\theta , \tag{3}$$

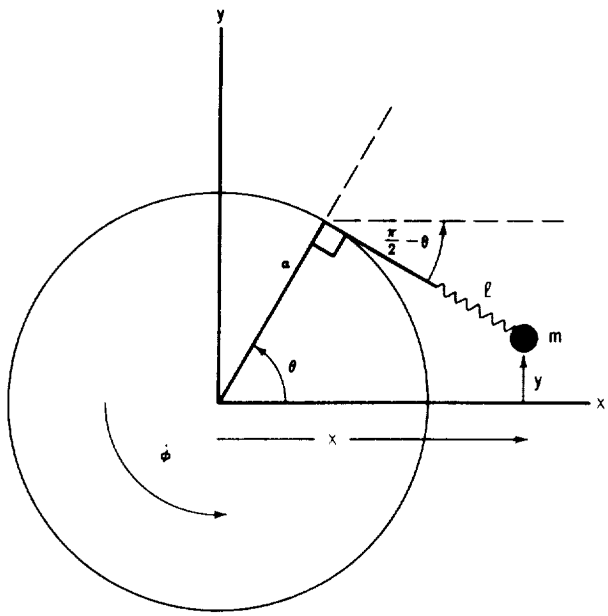

Figure 3—Phase 1 coordinate system.

and

$$l = a(\theta - \phi) + \delta, \tag{4}$$

where a is the radius of the satellite or de-spin fixture, $a(\theta - \phi)$ is the amount of unwrapped yo-yo, and δ is the stretch of the spring, the kinetic energy function can be put into the form

$$T = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \left(l^2 \dot{\theta}^2 + a^2 \dot{\phi}^2 + \dot{\delta}^2 - 2a\dot{\phi}\dot{\delta} \right).^* \tag{5}$$

The total potential energy of the system is $k\,\delta^2$ (two springs, where k is the spring constant). Thus the Lagrangian for the system is

$$L = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m \left(l^2 \dot{\theta}^2 + a^2 \dot{\phi}^2 + \dot{\delta}^2 - 2a\dot{\phi}\dot{\delta} \right) - k\,\delta^2 . \tag{6}$$

The equations of motion, in Lagrangian notation, are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = Q_\phi = 0 , \tag{7}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_\theta = 0 , \tag{8}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\delta}}\right) - \frac{\partial L}{\partial \delta} = Q_\delta = 0 . \tag{9}$$

Explicitly writing out the equations of motion results in:

$$\lambda^2 \frac{d\dot{\phi}}{dt} + a\, l\, \dot{\theta}^2 - a \frac{d^2\delta}{dt^2} = 0 , \tag{10}$$

$$\frac{d}{dt}\left(l^2 \dot{\theta}\right) - a\, l\, \dot{\theta}^2 = 0 , \tag{11}$$

*The dot indicates a derivative with respect to time and it is sometimes used with an actual time derivative for convenience of notation.

and

$$\frac{d^2\delta}{dt^2} + \frac{2k}{m}\delta - a\frac{d\dot{\phi}}{dt} - l\,\dot{\theta}^2 = 0 \ , \tag{12}$$

where $\lambda^2 = I/m + a^2$. Equations 10, 11, and 12 are a formidable set of simultaneous nonlinear differential equations, and it is unlikely that they can be solved analytically to give the spin reduction or the length of spring as a function of time as was done in Reference 2 for the rigid yo-yo. (The equations have been solved on an analog computer without much difficulty.)

Phase 2

A sketch of the Phase 2 situation is shown in Figure 4. The Lagrangian in this phase is

$$L = \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m\left[a^2\dot{\phi}^2 + l^2\dot{\gamma}^2 + \dot{l}^2 - 2a\,\dot{l}\,\dot{\phi}\sin(\phi-\gamma) + 2al\dot{\phi}\dot{\gamma}\cos(\phi-\gamma)\right] - k\,\delta^2 \ , \tag{13}$$

and the equations of motion are:

$$\frac{d}{dt}\left[\lambda^2\dot{\phi} - a\,\dot{l}\sin(\phi-\gamma) + a\,l\dot{\gamma}\cos(\phi-\gamma)\right] + a\,\dot{l}\dot{\phi}\cos(\phi-\gamma) + a\,l\,\dot{\phi}\dot{\gamma}\sin(\phi-\gamma) = 0 \ , \tag{14}$$

$$\frac{d}{dt}\left[l^2\dot{\gamma} + a\,l\,\dot{\phi}\cos(\phi-\gamma)\right] - a\,\dot{l}\dot{\phi}\cos(\phi-\gamma) - a\,l\,\dot{\phi}\dot{\gamma}\sin(\phi-\gamma) = 0 \ , \tag{15}$$

$$\frac{d}{dt}\left[\dot{\delta} - a\dot{\phi}\sin(\phi-\gamma)\right] - l\,\dot{\gamma}^2 - a\dot{\phi}\dot{\gamma}\cos(\phi-\gamma) + \frac{2k\delta}{m} = 0 \ , \tag{16}$$

where $l = l_0 + \delta$, l_0 being the unstretched length of the yo-yo. A brief glance at Equations 14, 15, and 16 shows that the equations of motion for Phase 2 are even more formidable than those for Phase 1.

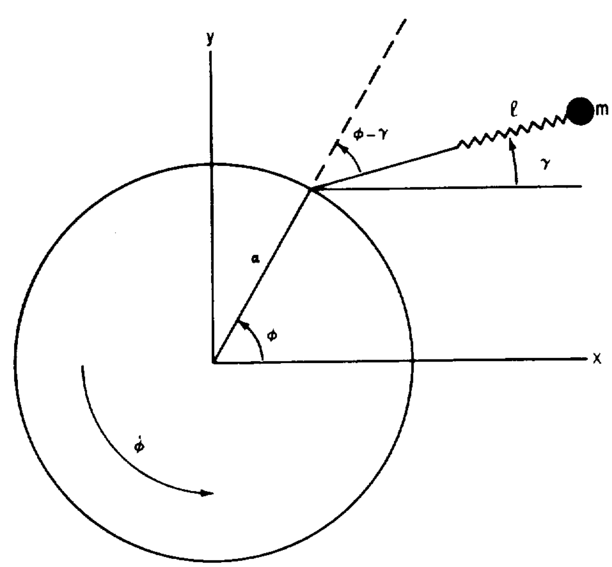

Figure 4—Phase 2 coordinate system.

SIMPLIFIED APPROACH TO THE STRETCH YO-YO DESIGN EQUATIONS

As was noted in earlier sections, the equations of motion for the stretch yo-yo are quite formidable and the direct equation of motion approach does not appear analytically fruitful. Equations for the conservation of momentum and energy, and a force equation will now be derived. A novel method of solution of these equations will then be applied. This will result in relatively simple design equations for the stretch yo-yo. Though they are not mathematically exact, analysis and tests have demonstrated that the equations are adequate for design and prediction of performance.

Combining Equations 14 and 15 of Phase 2 and integrating results in the equation for the conservation of momentum for Phase 2:

$$\lambda^2 \dot{\phi} - a \dot{l} \sin(\phi - \gamma) + a l \dot{\gamma} \cos(\phi - \gamma) + l^2 \dot{\gamma} + a l \dot{\phi} \cos(\phi - \gamma) = \text{constant} \quad . \tag{17}$$

At the release of the yo-yo $(\phi - \gamma = 0)$ Equation 17 reduces to

$$\frac{I}{m}\dot{\phi}_2 + (l + a)(a\dot{\phi}_2 + l\dot{\gamma}) = \text{constant}, \tag{18}$$

where $\dot{\phi}_2$ is the final spin rate in radians/second. If we neglect the oscillatory motion of the spring $(\dot{\delta})$, the conservation of energy equation at release is

$$\frac{1}{2} I \dot{\phi}_2^2 + \frac{1}{2} m (a\dot{\phi}_2 + l\dot{\gamma})^2 + k\delta^2 = \text{constant} \quad . \tag{19}$$

Equation 19 is essentially another integral of Equations 14, 15, and 16. A force equation can be obtained from Equation 16 by neglecting $\dot{\delta}$ and evaluating the equation at release; thus

$$k\delta = \frac{m}{2}(a\dot{\phi}_2^2 + l\dot{\gamma}^2) \quad . \tag{20}$$

A straightforward approach to solving Equations 18, 19, and 20 (that is, to eliminate γ and solve for the stretch and the spin mass of the system in terms of the other parameters) leaves one in a quagmire of algebraic complexity. What is desired is a yo-yo system where the final spin is completely independent of the initial spin. Analytical attempts to develop equations along these lines did not meet with success. It is possible though to develop simple analytical equations where, for small variations in the spring length, the effect on the final spin is negligible. We start the development by taking a variation of Equations 18 and 19 and letting $\Delta\dot{\phi}_2 = 0$:

$$(a\dot{\phi}_2 + l\dot{\gamma})\Delta\delta + (l + a)\Delta(a\dot{\phi}_2 + l\dot{\gamma}) = 0 \quad , \tag{21}$$

and

$$(a\dot{\phi}_2 + l\dot{\gamma})\Delta(a\dot{\phi}_2 + l\dot{\gamma}) + \frac{2k\delta}{m}\Delta\delta = 0 \quad . \tag{22}$$

Note that $\Delta l = \Delta\delta$. Dividing one equation by the other results in

$$(a\dot{\phi}_2 + l\dot{\gamma})^2 = 2\frac{k\delta}{m}(l + a) \quad . \tag{23}$$

Substituting Equation 23 into the energy equation (constant taken to be approximately $(I/2)\dot{\phi}_0^2$ where $\dot{\phi}_0$ is the initial spin rate in radians/second) and solving for the spring stretch δ gives

$$\delta = \frac{-(l_0 + a) + \sqrt{(l_0 + a)^2 + \frac{4I}{k}(\dot{\phi}_0^2 - \dot{\phi}_2^2)}}{4} \quad . \tag{24}$$

The positive sign is used in the quadratic formula since δ must be positive. By combining the momentum and energy equations and Equation 24 and using the approximation $\lambda^2 \approx I/m$, the following simple relationship connecting the spin mass with the initial and final spins can be obtained:

$$\frac{I(l + a + \delta)}{m(l + a)^3} = \frac{\dot{\phi}_0 + \dot{\phi}_2}{\dot{\phi}_0 - \dot{\phi}_2} = \frac{1 + r}{1 - r} \quad , \tag{25}$$

where $r = \dot{\phi}_2/\dot{\phi}_0$, the desired spin reduction. Note that Equation 25 is analogous to Equation 22 of Reference 2 for the rigid yo-yo. In fact, if δ is equal to zero in Equation 25 (rigid yo-yo), both equations are identical, as they should be.

To review the method briefly: essentially we had three simultaneous equations describing the stretch yo-yo system. What we did was to impose a physically desirable condition on two of the equations (conservation of energy and momentum) to replace the third equation. This enabled us to satisfy the conservation equations exactly and thus obtain an equation for the stretch of the yo-yo (spring) at its release without recourse to the additional equation. Also, an equation for the total spin mass was developed from the resulting equations. An estimate of how closely the third equation (force equation) is satisfied by this method is given in Appendix A.

The equations developed thus far do not put any restrictions on the spring constant k. We would like to use such a spring constant that for a variation in $\dot{\phi}_0$ and hence δ, the final variation in $\dot{\phi}_2$ would be approximately zero. A variation of Equation 24 gives

$$\Delta\delta = \frac{I\,\dot{\phi}_0\,\Delta\dot{\phi}_0}{k(l + a + 3\delta)} \quad . \tag{26}$$

A variation of Equation 25 gives

$$\left(\dot{\phi}_0^2 - \dot{\phi}_2^2\right)\left[\frac{l + a + 3\delta}{(l + a)(l + a + \delta)}\right]\Delta\delta = 2\dot{\phi}_2\,\Delta\dot{\phi}_0 \quad . \tag{27}$$

Combining Equations 26 and 27 gives the optimum k for the stretch yo-yo at design conditions,

$$k = \frac{\dot{\phi}_0^2\,(1 - r^2)I}{2r\,(l + a)(l + a + \delta)} \quad . \tag{28}$$

If the spring constant is the optimum one, then the stretch equation, Equation 24, reduces to

$$\delta = \frac{r(l_0 + a)}{1 - r} ; \tag{29}$$

and, in turn, Equation 28 simplifies to

$$k = \frac{\dot{\phi}_0^2 (1 - r)^3 I}{2r (l_0 + a)^2} , \tag{30}$$

and the mass equation, Equation 25, reduces to

$$\frac{I}{m(l + a)^2} = \frac{1}{1 - r} . \tag{31}$$

Thus for design conditions, the stretch yo-yo equations are expressed in terms of readily known quantities. Note in Equation 30 that if $r \to 0$, $k \to \infty$; that is, for zero final spin, the optimum stretch yo-yo becomes a rigid yo-yo. It should be emphasized that Equations 29, 30, and 31 are used only at design conditions. To calculate the stretch at any other spin (error computation for example) or k value, Equations 24 and 25 (or comparable preload equations) are used.

In the practical world of fabrication, parts (for example, the stretch yo-yo springs) are not made exactly the way equations specify. Springs are not made precisely with a spring constant of say 12.5 lb/ft; there is a tolerance about this value. Also, springs can have a preload which the derived equations do not take into account. By following a procedure similar to what has been outlined, a stretch equation taking preload into account can be derived:

$$\delta = \frac{-\left(l_0 + a + \frac{3F_0}{k}\right) + \sqrt{\left(l_0 + a + \frac{3F_0}{k}\right)^2 + 4\left[\frac{I}{k}\left(\dot{\phi}_0^2 - \dot{\phi}_2^2\right) - \frac{2F_0}{k}\left(l_0 + a\right)\right]}}{4} , \tag{32}$$

where F_0 is the spring preload. Also, a mass equation can be developed:

$$\frac{I\left(l + a + \delta + \frac{\delta F_0}{k\delta + F_0}\right)}{m(l + a)^3} = \frac{\dot{\phi}_0 + \dot{\phi}_2}{\dot{\phi}_0 - \dot{\phi}_2} = \frac{1 + r}{1 - r} . \tag{33}$$

These equations are used after hardware has been fabricated and a new mass weight must be calculated to correct for deviations from the optimum spring constant and preload. A desk calculator should be used in evaluating Equations 32 and 33 for accurate stretch yo-yo results. Calculations have indicated and tests have supported the fact that the preload of the spring is beneficial. That is, the spin error is less with preloaded springs. Full advantage cannot be taken of this because spring preload is an unpredictable thing.

An important element in the stretch yo-yo is the spring. Design procedures for the spring are developed in Appendix B.

APPLICATION OF DESIGN EQUATIONS AND DISCUSSION

We shall now apply the equations to a practical situation to illustrate how the equations are used and what precautions should be taken. The experimental verification of the stretch yo-yo equations will be covered in a future NASA Technical Note.

The following quantities are involved in establishing a stretch yo-yo design: I, a, $\dot{\phi}_0$, $\dot{\phi}_2$, l_0, δ, k, and m. The first four quantities are usually specified; l_0 is at the discretion of the designer; the last three quantities can be calculated from the developed equations. As an example, consider the design of a stretch yo-yo for a typical Goddard Space Flight Center satellite. Initial design parameters are tabulated below:

$I = 2.885$ slug-ft^2,

$a = 0.942$ ft,

$\dot{\phi}_0 = 160$ rpm $= 16.755$ rad/sec,

$\dot{\phi}_2 = 73.9$ rpm $= 7.735$ rad/sec.

Therefore $r = 0.462$. The satellite structure is such that only about a half wrap of yo-yo will fit. Hence l_0 is chosen to be 2.365 ft. The design spin reduction stretch δ is calculated from Equation 29,

$$\delta = \frac{r\left(l_0 + a\right)}{1 - r} = 2.841 \text{ ft} .$$

Thus, the total length of one yo-yo at release is $l = l_0 + \delta = 5.206$ ft. The optimum value for the spring constant is calculated from Equation 30,

$$k = \frac{\dot{\phi}_0^2 (1 - r)^3 I}{2r (l_0 + a)^2} = 12.48 \text{ lb/ft} .$$

The force in the spring (wire) at design conditions is $k\delta = (12.48)\ 2.841 = 35.46$ pounds. The total spin weight (mass) of the system is obtained from Equation 31,

$$m = \frac{I (1 - r)}{(l + a)^2} = 0.0411 \text{ slugs} = 600.5 \text{ gm}.$$

For the spring analysis, the procedure described in Appendix B is used. The quantities listed below were chosen (or were true for the selected spring material):

wire = NS355 spring steel,

ρ = weight density of the spring = 0.282 lb/in.^{3},

G = shear modulus of elasticity = 11.5×10^6 lb/in.^{2} ,

l_s = length of spring = 25 in.,

d = wire diameter = 0.0625 in.

The following quantities were calculated according to the procedure in Appendix B:

R = mean helix radius = 0.1875 in.,

Coil OD = 0.4375 in.,

C = spring index = $\frac{2R}{d}$ = 6.0,

k_s = stress concentration factor for torsion and transverse shear = 1.088,

F_{max} = $k\,\delta_{max}$ = 49 lb (25 percent overspin),

S_{max} = maximum torsional stress in the spring = 208,500 lb/in.^{2},

m_s = weight of spring = 0.4077 lb = 185.1 gm.

Static tests of the spring showed that the 49 pound load was the maximum that the spring could maintain and still operate in the linear region.

Reference 2 shows that the mass of the wire in a rigid yo-yo can be approximately accounted for by adding 1/3 the mass of the wire to the spin mass. Full scale tests have verified that this method is also applicable to the stretch yo-yo. We thus define the total mass m_t of the yo-yo system in the following way:

$$m_t = 2\left(m_0 + \frac{m_s + m_w}{3}\right), \tag{34}$$

where

m_0 = mass of one spin weight,

m_s = mass of one spring,

m_w = mass of one wire (if there is any).

Once the weight of the spring has been established the weight of the end mass can be calculated from Equation 34:

$$m_0 = \frac{m_t}{2} - \frac{m_s}{3} = 238.6 \text{ gm}.$$

This completes the preliminary design calculations.

The error curve expected from this design is shown in Figure 5 (see Appendix C for the error calculation procedure). It will be noticed from Figure 5 that for ±20 percent error in initial spin, the final de-spin value is less than 5 percent from the desired spin. In contrast, for the rigid yo-yo, a 20 percent error in initial spin would result in a 20 percent error in the final spin. Note also in Figure 5 the flat portion of the error curve near zero percent spin-up error. This shows that the physically desired condition has been successfully imposed on the stretch yo-yo equations. That is, small spin-up errors have negligible effect on the final spin.

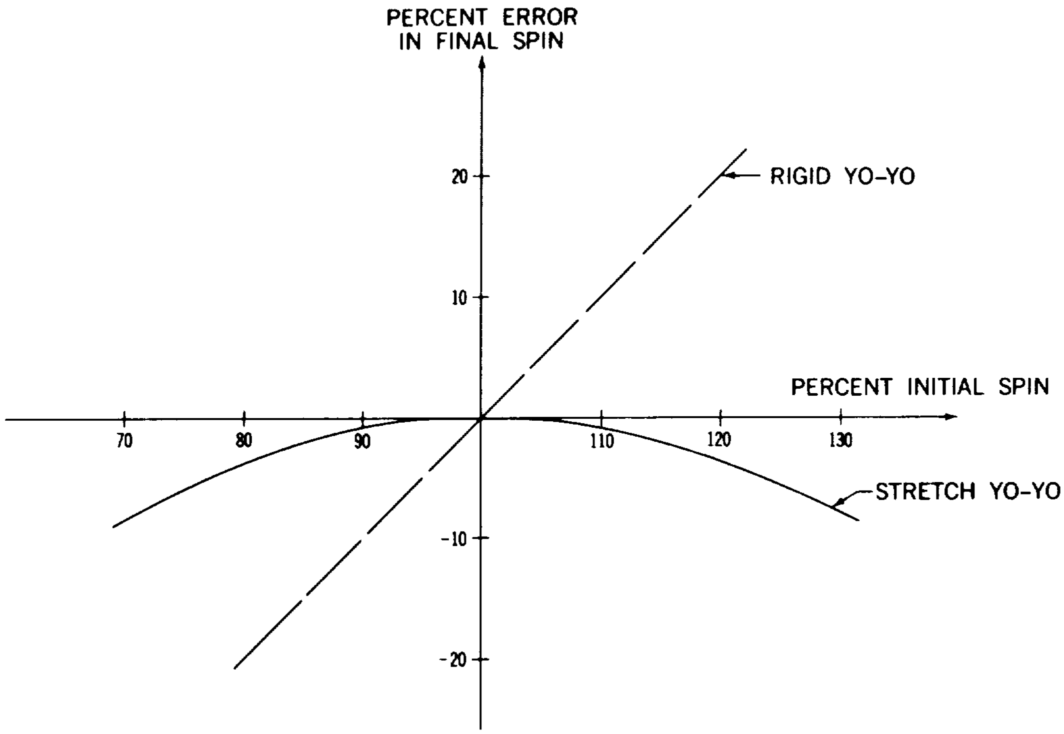

Figure 5—Final spin error vs. spin-up error of a stretch yo-yo compared to that of a rigid yo-yo.

Figure 6 shows an error curve when the spin moment of inertia is varied ±20 percent. Superimposed is the error curve for a comparable rigid yo-yo. Except for the fortuitous conditions of a 20 percent increase in moment of inertia and a 20 percent decrease in initial spin-up (or vice versa), the stretch yo-yo is clearly less sensitive to variations in the spin moment of inertia. Physically, the error correction can be explained as follows: for design spin moment of inertia there is a certain amount of stretching of the spring during normal operation. If the spin moment of inertia is greater than the design value, the spin will be higher during early periods of the de-spin cycle (as compared to normal operation) because of the greater kinetic energy. This causes the stretch yo-yo to elongate more and thus compensate for the increased kinetic energy in the system. A similar explanation holds for a moment of inertia less than the design value. This type of action thus tends to reduce variations from the desired final spin.

As was pointed out in an earlier section, when the hardware is fabricated and accurate values of the spring constant, spring weight, and preload are known, Equations 32 and 33 must be used to calculate a precise value for the spin weight. It is also advantageous to have a plot of the spin moment of inertia versus the spin weight so that last minute changes due to increased satellite weight can be included. For convenience in making design calculations, a computation sheet is included in Appendix D.

During the writing of this technical note, a stretch yo-yo was used on Ariel I (1962 $o1$), April 1962.

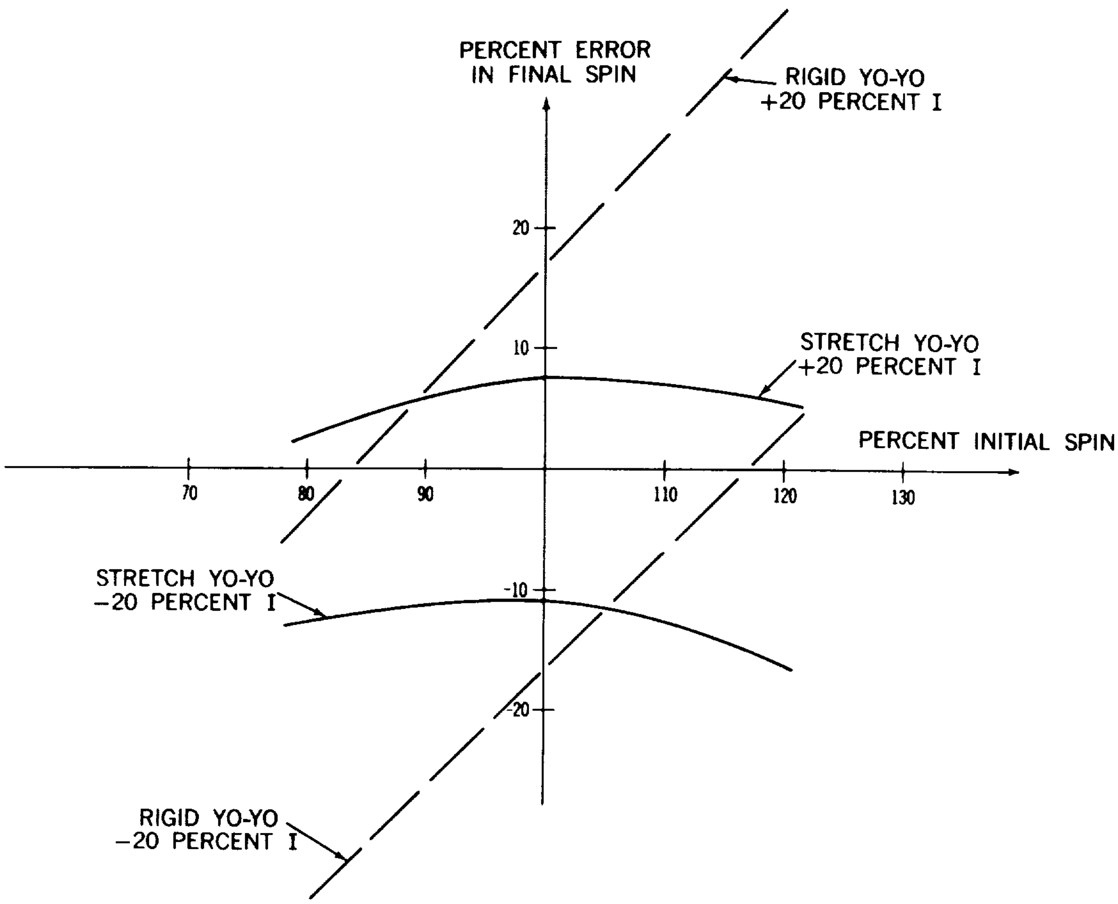

Figure 6—Final spin error vs. spin-up error of a stretch yo-yo compared to that of a rigid yo-yo (spin moment of inertia varied ±20 percent).

RESUMÉ

The equations for an advanced type of yo-yo for satellite de-spin have been developed. As was noted, by using a variational method of analysis, simple design equations have been derived. A straightforward step by step procedure has been obtained for the design of the stretch yo-yo. The results calculated from the equations clearly indicate that the stretch yo-yo is less sensitive to satellite spin-up errors and uncertainty in the spin moment of inertia than the rigid yo-yo.

ACKNOWLEDGMENTS

The author is indebted to W. R. Mentzer, Jr. of Goddard Space Flight Center for checking the equations and developing Appendix B, and for the numerous calculations made prior to flying a stretch yo-yo.

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Appendix A

Error Estimate for the Force Equation

In the variational approach, a physically desirable condition was imposed on the conservation of energy and momentum equations. The resulting equation was used in place of the force equation to derive the stretch yo-yo properties. It is the purpose of this section to show what is neglected in the force equation when the variational method is used, and to give an indication of the accuracy of the derived equations.

The force equation (Equation 20 of the body of this report) is:

$$k\,\delta = \frac{m}{2}\left(a\,\dot{\phi}_2^2 + l\,\dot{\gamma}^2\right) \quad . \tag{A1}$$

Multiplying Equation A1 by $l + a$ and rearranging gives

$$(l + a)\,k\,\delta = \frac{m}{2}\left(a^2\,\dot{\phi}_2^2 + l^2\,\dot{\gamma}^2\right) + \frac{ma\,l}{2}\left(\dot{\phi}_2^2 + \dot{\gamma}^2\right) . \tag{A2}$$

This can be put into the form,

$$(l + a)\,k\,\delta = \frac{m}{2}\left(a\,\dot{\phi}_2 + l\,\dot{\gamma}\right)^2 + \frac{ma\,l}{2}\left(\dot{\gamma} - \dot{\phi}_2\right)^2 \quad , \tag{A3}$$

by adding and subtracting $m\,a\,l\,\dot{\phi}\,\dot{\gamma}$. From the energy equation (Equation 19 of the report proper) we have

$$\frac{1}{2}m\left(a\,\dot{\phi}_2 + l\,\dot{\gamma}\right)^2 = \frac{1}{2}I\left(\dot{\phi}_0^2 - \dot{\phi}_2^2\right) - 2\,k\,\delta^2 \quad . \tag{A4}$$

Substituting Equation A4 into Equation A3 and rearranging gives

$$I\left(\dot{\phi}_0^2 - \dot{\phi}_2^2\right) = 2\,k\,\delta^2 + 2\,(l + a)\,k\,\delta - ma\,l\left(\dot{\gamma} - \dot{\phi}_2\right)^2 \quad . \tag{A5}$$

Compare this with Equation 24 of the body of the report, which was obtained from the variational method and has been squared and arranged:

$$I\left(\dot{\phi}_0^2 - \dot{\phi}_2^2\right) = 2\,k\,\delta^2 + 2\,(l + a)\,k\,\delta \quad . \tag{A6}$$

We see that $ma\,l\,(\dot{\gamma} - \dot{\phi}_2)^2$ has been neglected in Equation A6. Hence, in the variational method of analysis we are neglecting $ma\,l\,(\dot{\gamma} - \dot{\phi}_2)^2$ compared to $2(l + a)k\delta + 2k\delta^2$ or $I(\dot{\phi}_0^2 - \dot{\phi}_2^2)$. Defining the ratio of these as E, we have

$$E = \frac{m a\, l\left(\dot{\gamma} - \dot{\phi}_2\right)^2}{I\left(\dot{\phi}_0^2 - \dot{\phi}_2^2\right)} . \tag{A7}$$

An estimate of $\dot{\gamma} - \dot{\phi}_2$ must be made so that E can be evaluated. Letting $\dot{\gamma} - \dot{\phi}_2 = \epsilon$, substituting this into the momentum equation (Equation 18 of the body of this report), and rearranging and comparing the result with Equation 25, we can conclude that $l/(l + a)\,\epsilon$ must be equal to

$$\frac{\dot{\phi}_0\,(l + a - \delta r)}{(l + a + \delta)} ,$$

for the two equations to be identical. Since $\delta = r(l + a)$, for design conditions, letting $l/(l + a) \approx 1$ we find

$$\epsilon = \dot{\phi}_0 - \dot{\phi}_2 . \tag{A8}$$

If we substitute Equations A8 and 31 into Equation A7, E can be expressed as

$$E = \frac{(1 - r)^2\, a\, l}{(l + a)^2\,(1 + r)} . \tag{A9}$$

A calculation of E for the application mentioned in the discussion (see the body of the report) gives

$$E = \frac{(0.538)^2\,(0.942)\,(5.206)}{(6.148)^2\,(1.462)} = 0.0256 .$$

This indicates that the neglected term is approximately 1/40th of the retained terms. Since in general a square root is taken to determine δ, the error in δ should be even less. Since the conservation of energy and momentum equations are satisfied exactly (within the approximation $I/m >> a^2$), the implication is that for engineering purposes the variational method gives essentially exact results. Of course in any design it is prudent to calculate E to verify that the neglected term is indeed small.

Appendix B

Spring Design

An important element in the stretch yo-yo is the spring. Conventional music wire can be used for the spring in some applications, but the most satisfactory spring material has been that manufactured by the National Standard Company of Niles, Michigan. This material has an appreciably higher tensile strength than conventional music wire.

For predictable yo-yo performance, the spring should operate in the linear region. Hence the torsional stress level in the spring must be checked. Stress equations and spring design criteria given by Spotts may be used.* There are other methods that can be used, but the following seems to work well.

The maximum torsional stress in the spring is

$$S_{max} = \frac{16}{\pi} F_{max} k_s \frac{R}{d^3} , \tag{B1}$$

where

F_{max} = $k \delta_{max}$ (lb),

k_s = stress concentration factor for torsion and transverse shear,

R = mean helix radius (in.),

d = wire diameter (in.).

S_{max} should be within the torsional yield stress prescribed by Spotts.* Thus

$$S_{yp\ torsion} = 0.6\ S_{yp\ tension} . \tag{B2}$$

The other related spring equations are

$$\frac{R}{d^3} = \sqrt[3]{\frac{G}{64 k\, l_s\, d^4}} , \tag{B3}$$

*Spotts, M. F., "Design of Machine Elements," Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1960.

and

$$m_s = \frac{G \rho \pi^2}{128\, k \left(\frac{R}{d^3}\right)^2}, \tag{B4}$$

where

G = shear modulus of elasticity (lb/in.^{2}),

k = spring constant (lb/in.),

l_s = length of spring (close wound) (in.),

m_s = weight of spring (lb),

ρ = weight density of spring (lb/in.^{3}).

Spring design procedure:

1. Select a spring length, wire size, and type (l_s, d, G, ρ).
2. Compute R/d^3 from Equation B3; R can then be found.
3. Compute the spring index $C = 2R/d$ and find the stress concentration factor k_s from Figure 4-4 of Spotts' book.* The value for k_s is usually between 1.0 and 1.2.
4. Compute F_{max} ($= k\delta$).
5. Compute S_{max} (Equation B1) and check if it is within the torsional stress limit of the material (Equation B2). If not, select a new wire diameter and repeat the above procedure;
6. If the stress level is satisfactory, compute the mass of the spring from Equation B4.

Appendix C

Error Curve Procedure

The error curve can be obtained in the following way. Choose a new spin-up value (for example, 20 percent over design spin-up); calculate the yo-yo stretch from Equation 24 assuming $\dot{\phi}_2$ is the design final spin. Check this assumption by calculating the final spin from Equation 25. This new value of $\dot{\phi}_2$ can be used in Equation 24 to calculate a new yo-yo stretch, etc. Calculations performed indicate that final spin does not change much and the first calculation for $\dot{\phi}_2$ is usually 99.9 percent of the second calculated value of $\dot{\phi}_2$.

*Spotts, M. F., "Design of Machine Elements," Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1960.

Appendix D

Stretch Yo-Yo De-Spin Calculation Sheet
(Radial Release and Design Conditions Only)

Definitions of Symbols and Units:

I - moment of inertia about spin axis (slug-ft^2),
a - radius of de-spin fixture (ft),
l_0 - unstretched length of one yo-yo (ft),
δ - stretch of one yo-yo at release (ft),
k - spring constant (lb/ft),
m_t - total mass of yo-yo system (slugs),
m_s - mass of one spring (slugs),
m_w - mass of one wire (slugs),
m_0 - mass of one spin weight (slugs),
$\dot{\phi}_0$ - initial spin rate (rad/sec),
$\dot{\phi}_2$ - final spin rate (rad/sec).

Record:

I = ________ slug-ft^2,
a = ________ ft,
$\dot{\phi}_0$ = ________ rad/sec,
$\dot{\phi}_2$ = ________ rad/sec,
l_0 = ________ ft,
m_s = ________ slugs.

Calculate:

$$r = \frac{\dot{\phi}_2}{\dot{\phi}_0} = \_\_\_\_\_\_ = \_\_\_\_\_\_ ;$$

Stretch at release,

$$\delta = \frac{r(l_0 + a)}{1 - r} = \_\_\_\_\_\_ = \_\_\_\_\_\_ \text{ ft};$$

Optimum spring constant,

$$k = \frac{\dot{\phi}_0^2 (1 - r)^3 I}{2r (l_0 + a)} = \_\_\_\_\_\_ = \_\_\_\_\_\_ \text{ lb/ft};$$

Force at release,

$$\text{force} = k\,\delta = \_\_\_\_\_\_ = \_\_\_\_\_\_ \text{ lb};$$

Total spin mass (weight) of yo-yo system,

$$m_t = \frac{I(1 - r)}{(l_0 + \delta + a)^2} = \_\_\_\_\_\_ = \_\_\_\_\_\_ \text{ slugs},$$

$$= \_\_\_\_\_\_ \text{ gm};$$

Single spin mass*

$$m_0 = \frac{m_t}{2} - \frac{(m_s + m_w)}{3} = \_\_\_\_\_\_ = \_\_\_\_\_\_ \text{ slugs},$$

$$= \_\_\_\_\_\_ \text{ gm}.$$

*See Appendix B for calculation of the spring mass (m_s).

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